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Abstract

Let F_N be a free group of finite rank $N \geq 2$, and let T be an $\mathbb{R}$-tree with a very small, minimal action of F_N with dense orbits. For any basis $\mathcal{A}$ of F_N there exists a *heart* $K_\mathcal{A} \subset \overline{T}$ (= the metric completion of T) which is a compact subtree that has the property that the dynamical system of partial isometries $a_i : K_\mathcal{A} \cap a_i K_\mathcal{A} \to a_i^{-1} K_\mathcal{A} \cap K_\mathcal{A}$, for each $a_i \in \mathcal{A}$, defines a tree $T_{(K_\mathcal{A}, \mathcal{A})}$ which contains an isometric copy of T as minimal subtree.

1 Introduction

A point on Thurston's boundary of Teichmüller space $\mathcal{T}(\Sigma)$ for a surface Σ can be understood alternatively as a measured geodesic lamination $(\mathfrak{L}, \mu)$ on Σ, up to rescaling of the transverse measure, or as a small action of $\pi_1\Sigma$ on some $\mathbb{R}$-tree T, up to $\pi_1\Sigma$-equivariant homothety. The correspondence between these two objects, which are naturally dual to each other, is given by the fact that points of T are in 1-1 correspondence (or "one-to-finite" correspondence, for the branchpoints of T) with the leaves of $\widetilde{\mathfrak{L}}$, i.e. the lift of $\mathfrak{L}$ to the universal covering $\widetilde{\Sigma}$. The metric on T is determined by μ, and vice versa.

Culler-Vogtmann's Outer space CV_N is the analogue of $\mathcal{T}(\Sigma)$, with $\mathrm{Out}(F_N)$ replacing the mapping class group. A point of the boundary $\partial\mathrm{CV}_N$ is given by a homothety class $[T]$ of very small isometric actions of the free group F_N on an $\mathbb{R}$-tree T. In general, T will not be dual to a measured lamination on a surface. However, in [CHL-I, CHL-II] an "abstract" *dual lamination* $L(T)$ has been defined for any such T, which is very much the analogue of $\mathfrak{L}$ in the surface case. The dual lamination $L(T)$ is an algebraic lamination: it lives in the double Gromov boundary of F_N, and the choice of a basis $\mathcal{A}$ transforms $L(T)$ into a symbolic dynamical system which is a classical subshift in $\mathcal{A} \cup \mathcal{A}^{-1}$. The dual lamination $L(T)$, and variations of it, have already been proved to be a useful invariant of the tree T, compare [BFH00, CHL05, HM06, KL07].

In the case of measured laminations on a surface, the standard tool which allows a transition from geometry to combinatorial dynamics, is given by interval exchange transformations. The combinatorics which occur here are classically given through coding geodesics on a surface by sequences of symbols, where the symbols correspond to subintervals, and the sequences are given by the first return map. Conversely, the surface and the lamination (or rather "foliation", in this case), can be recovered from the interval exchange transformation by suspension, i.e. by realizing the map which exchanges the subintervals by a (foliated) mapping torus.

Taking the basic concept of this classical method one step further and considering directly the dual tree T rather than the lamination given by the combinatorial data, one considers for any $[T] \in \partial\mathrm{CV}_N$ a finite metric subtree $K \subset T$, and for some basis $\mathcal{A}$ of F_N the induced finite system of partial isometries between subtrees of K: Each basis element $a_i \in \mathcal{A}$ defines a partial isomerty $a_i : K \cap a_i K \to a_i^{-1} K \cap K$, and these partial isometries play the role of the interval exchange transformation. Any such pair $\mathcal{K} = (K, \mathcal{A})$ gives canonically rise to a tree $T_{\mathcal{K}}$ together with an F_N-equivariant map $j : T_{\mathcal{K}} \to T$. The tree $T_{\mathcal{K}}$ is the "unfolding space" of the system $\mathcal{K}$. The class of $\mathbb{R}$-trees T, with the property that for some such finite K the map j is an isometry, have been investigated intensely, and they play an important role in the study of $\partial\mathrm{CV}_N$, see [GL95].

Indeed, if K is an interval and if it is simultaneously equal to the union of domains and the union of ranges of the isometries (and if these unions are disjoint unions except at the boundary points), then $\mathcal{K}$ defines actually an interval exchange transformation. If one only assumes that K is finite, this will in general not be true: one only obtains a system of interval translations (see for instance [BH04]). On the level of $\mathbb{R}$-trees one obtains in the first case *surface tree actions*, and in the second case actions that were alternatively termed *Levitt, thin* or *exotic*. The union of these two classes are precisely the actions called *geometric* in [GL95].

However, both of these types of actions seem to be more the exception than the rule: Given any point $[T] \in \partial\mathrm{CV}_N$, there is in general no reason why T should be determined by a system of partial isometries based on a finite tree $K \subset T$. A possible way to deal with such T is to consider increasing sequences of finite subtrees and thus to approximate T by the sequence of ensuing geometric trees $T_{\mathcal{K}}$, in the spirit of the "Rips machine", which is an important tool to analyze arbitrary group actions on $\mathbb{R}$-trees. The goal of this paper is to propose a more direct alternative to this approximation technology:

We replace the condition on the subtree $K \subset T$ to be finite by the weaker condition that K be compact. It turns out that almost all of the classical machinery developed for the approximation trees $T_{\mathcal{K}}$ for finite K carries over directly to the case of compact K. However, the applications of such $T_{\mathcal{K}}$ concern a much larger class of trees: In particular, every minimal very small T with dense orbits can be described directly, i.e. circumventing completely the above approximation, as minimal subtree $T_{\mathcal{K}}^{min}$ of the tree $T_{\mathcal{K}}$, for a properly chosen compact subtree K of the metric completion $\overline{T}$ of T.

Theorem 1.1. *Let T be an $\mathbb{R}$-tree provided with a very small, minimal, isometric action of the free group F_N with dense orbits. Let $\mathcal{A}$ be a basis of F_N. Then there exists a unique compact subtree $K_{\mathcal{A}} \subset \overline{T}$ (called the "heart" of T w.r.t. $\mathcal{A}$), such that for any compact subtree K of $\overline{T}$ one has:*

$$T = T_{\mathcal{K}}^{min} \iff K_{\mathcal{A}} \subseteq K$$

This is a slightly simplified version of Theorem 5.4 proved in this paper. The main tool for this proof (and indeed for the definition of the heart $K_{\mathcal{A}}$) is the dual lamination $L(T)$. We define in this article (see §3) a second *admissible lamination* $L_{\text{adm}}(\mathcal{K})$ associated to the system of partial isometries $\mathcal{K} = (K, \mathcal{A})$. One key ingredient in the equivalence of Theorem 1.1 is to prove that the two statements given there are equivalent to the equation $L(T) = L_{\text{adm}}(\mathcal{K})$. The other key ingredient, developed in §4, is a new understanding of the crucial map $\mathcal{Q} : \partial F_N \to \overline{T} \cup \partial T$ from [LL03], based on the dynamical system $\mathcal{K} = (K, \mathcal{A})$. The proof of Theorem 5.4 uses the full strength of the duality between trees and laminations, and in particular a transition between the two, given by the main result of our earlier paper [CHL05].

We would like to emphasize that the main object of this paper, the heart $K_{\mathcal{A}}$ of T with respect to any basis $\mathcal{A}$ of F_N, is a compact subtree of $\overline{T}$ that is determined by algebraic data associated to T, namely by the dual algebraic lamination $L(T)$ of T. This system $\mathcal{K}_{\mathcal{A}} = (K_{\mathcal{A}}, \mathcal{A})$ of partial isometries is entirely determined by the choice of the basis $\mathcal{A}$ and it depends on $\mathcal{A}$, but important properties of it turn out to be independent of that choice. For example, we derive in §6 from the above theorem the following direct characterization of geometric trees, and we also give a sharpening of Gaboriau-Levitt's approximation result for trees T from $\partial \mathrm{CV}_N$:

Corollary 6.1. *A very small minimal $\mathbb{R}$-tree T, with isometric F_N-action that has dense orbits, is geometric if and only if, for any basis $\mathcal{A}$ of F_N, the heart $K_{\mathcal{A}}$ is a finite subtree of T.*

Corollary 6.3. *For every very small minimal $\mathbb{R}$-tree T, with isometric F_N-action that has dense orbits, there exists a sequence of finite subtrees $K(n)$ of uniformely bounded diameter, such that:*

$$T = \lim_{n \to \infty} T_{\mathcal{K}(n)}$$

In contrast to the case of geometric $\mathbb{R}$-trees, there are trees in $\partial \mathrm{CV}_N$ for which the compact heart is far from being finite. Indeed it is proven in [Cou08] that the compact heart of the repulsive tree $T_{\Phi^{-1}}$ of an iwip outer automorphism Φ of F_N has Hausdorff dimension equal to $\max(1, \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}})$, where λ_Φ and $\lambda_{\Phi^{-1}}$ are the expansion factors of Φ and Φ^{-1} respectively. As these expansion factors are in general not equal, we can assume that $\lambda_\Phi > \lambda_{\Phi^{-1}}$ to get a compact heart with Hausdorff dimension strictly bigger than 1.

2 F_N-actions on $\mathbb{R}$-trees and their heart

In this section we first recall some well known facts about $\mathbb{R}$-trees T with isometric action of a free group F_N. We also recall algebraic laminations, and in particular the dual lamination $L(T)$. We then concentrate on the specific case of very small trees with dense orbits, and for such trees we define the limit set and the heart of T with respect to a fixed basis $\mathcal{A}$ of F_N.

In this paper we need some of the machinary developed in our previous articles [CHL-I, CHL-II, CHL05]. We present these tools in this section, but refer to those articles for proofs and for a more complete discussion.

2.1 Background on $\mathbb{R}$-trees

An $\mathbb{R}$-tree T is a metric space which is 0-hyperbolic and geodesic. Alternatively, a metric space T is an $\mathbb{R}$-tree if and only if any two points $x, y \in T$ are joined by a unique topological arc $[x, y] \subset T$, and this arc (called a *segment*) is geodesic. For any $\mathbb{R}$-tree T, we denote by $\overline{T}$ the metric completion and by ∂T the Gromov boundary of T. We also write $\widehat{T} = \overline{T} \cup \partial T$.

Most $\mathbb{R}$-trees T considered in this paper are provided with an action by isometries (from the left) of a non-abelian free group F_N of finite rank $N \geq 2$. Such an action is called *minimal* if T agrees with its minimal F_N-invariant subtree. We say that the action has *dense orbits* if for some (and hence every) point $x \in T$ the orbit $F_N \cdot x$ is dense in T. In the case of dense orbits, the following three conditions are equivalent:

- T has *trivial arc stabilizers* (i.e. for any distinct $x, y \in T$ and $w \in F_N$ the equality $w[x, y] = [x, y]$ implies $w = 1$).
- The F_N-action on T is *small* (see [CM87, CHL-II]).
- The F_N-action on T is *very small* (see [CL95, CHL-II]).

As usual, for any $w \in F_N$ we denote by $\|w\|_T$ (or simply by $\|w\|$) the translation length of the action of w on T, i.e. the infimum of $d(x, wx)$ over all $x \in T$.

There are two types of isometries of T: An element $w \in F_N$ acts as an *elliptic* isometry on T if it fixes a point, which is equivalent to $\|w\| = 0$. If $\|w\| > 0$, then the action of w on T is called *hyperbolic*: There is a well defined *axis* in T, which is isometric to $\mathbb{R}$ and is w-invariant: the element w translates every point on the axis by $\|w\|$.

A continuous map $T \to T'$ between $\mathbb{R}$-trees is called a *morphism* if every segment is mapped locally injectively except at finitely many points.

2.2 The observers' topology on T

There are various independent approaches in the literature to define $\mathbb{R}$-trees as topological spaces without reference to the metric. The following version has been studied in [CHL05].

Definition 2.1. Let T be an $\mathbb{R}$-tree. A *direction* in $\widehat{T}$ is a connected component of the complement of a point of $\widehat{T}$. A subbasis of open sets for the *observers' topology* on $\widehat{T}$ is given by the set of all such directions in $\widehat{T}$.

The observers' topology on $\widehat{T}$ (or T) is weaker than the metric topology: For example, any sequence of points that "turns around" a branch point converges to this branch point. We denote by $\widehat{T}^{\text{obs}}$ the set $\widehat{T}$ equipped with the observers' topology. The space $\widehat{T}^{\text{obs}}$ is Hausdorff and compact.

For any sequence of points P_n in $\widehat{T}$, and for some *base point* $Q \in \widehat{T}$, there is a well defined *inferior limit from* Q, which we denote by:

$$P = \liminf_{n\to\infty}{}_Q \ P_n$$

It is given by

$$[Q, P] = \overline{\bigcup_{m=0}^{\infty} \bigcap_{n\geq m} [Q, P_n]}.$$

The inferior limit P is always contained in the closure of the convex hull of the P_n, but its precise location does in fact depend on the choice of the base point Q. However, in [CHL05] the following has been shown:

Lemma 2.2. *If a sequence of points P_n converges in $\widehat{T}^{obs}$ to some limit point $P \in \widehat{T}^{obs}$, then for any $Q \in \widehat{T}$ one has:*

$$P = \liminf_{n\to\infty}{}_Q \ P_n$$

The observers' topology is very useful, but it is also easy to be deceived by it. For example, it is not true that any continuous map between $\mathbb{R}$-trees $T_1 \to T_2$ induces canonically a continuous map $\widehat{T}_1^{\text{obs}} \to \widehat{T}_2^{\text{obs}}$, as is illustrated in the following remark.

Remark 2.3. Let T_1 be the ∞-pod, given by a center Q and edges $[Q, P_k]$ of length 1, for every $k \in \mathbb{N}$. Let T_2 be obtained from T_1 by gluing the initial segment of length $\frac{k-1}{k}$ of each $[Q, P_k]$, for $k \geq 2$, to $[Q, P_1]$. Then the canonical map $f : T_1 \to T_2$ is continuous, and even a length decreasing morphism, but $\lim P_k = Q$, while $\lim f(P_k) = f(P_1) \neq f(Q)$.

We refer the reader to [CHL05] for more details about the observers' topology.

2.3 Algebraic laminations

For the free group F_N of finite rank $N \geq 2$, we denote by ∂F_N the Gromov boundary of F_N. We also consider

$$\partial^2 F_N = \partial F_N \times \partial F_N \smallsetminus \Delta\,,$$

where Δ denotes the diagonal. The space $\partial^2 F_N$ inherits from ∂F_N a left-action of F_N, defined by $w(X,Y) = (wX, wY)$ and a topology. It also admits the *flip map* $(X,Y) \mapsto (Y,X)$. An *algebraic lamination* $L^2 \subset \partial^2 F_N$ is a non-empty closed subset which is invariant under the F_N-action and the flip map.

If one choses a basis $\mathcal{A}$ of F_N, then every element $w \in F_N$ can be uniquely written as a finite reduced word in $\mathcal{A}^{\pm 1}$, so that F_N is canonically identified with the set $F(\mathcal{A})$ of such words. Similarly, a point of the boundary ∂F_N can be written as an infinite reduced word $X = z_1 z_2 \ldots$, so that ∂F_N is canonically identified with the set $\partial F(\mathcal{A})$ of such infinite words.

We also consider reduced biinfinite indexed words

$$Z = \ldots z_{-1} z_0 z_1 \ldots$$

with all $z_i \in \mathcal{A}^{\pm 1}$. We say that Z has *positive half* $Z^+ = z_1 z_2 \ldots$ and *negative half* $Z^- = z_0^{-1} z_{-1}^{-1} \ldots$, which are two infinite words

$$Z^+, Z^- \in \partial F(\mathcal{A})$$

with distinct initial letters $Z_1^+ \neq Z_1^-$. We write the reduced product $Z = (Z^-)^{-1} \cdot Z^+$ to mark the letter Z_1^+ with index 1.

For any fixed choice of a basis $\mathcal{A}$, an algebraic lamination L^2 determines a *symbolic lamination*

$$L_{\mathcal{A}} = \{(Z^-)^{-1} \cdot Z^+ \mid (wZ^-, wZ^+) \in L^2\}$$

as well as a *laminary language*

$$\mathcal{L}_{\mathcal{A}} = \{w \in F(\mathcal{A}) \mid w \text{ is a subword of some } Z \in L_{\mathcal{A}}\}\,.$$

Both, symbolic laminations and laminary languages can be characterized independently, and the natural transition from one to the other and back to an algebraic lamination has been established with care in [CHL-I]. In case we do not want to specify which of the three equivalent terminologies is meant, we simply speak of a *lamination* and denote it by L.

One of the crucial points of the encounter between symbolic dynamics and geometric group theory, in the subject treated in this paper, occurs precisely at the transition between algebraic and symbolic laminations. Since the main thrust of this paper (as presented in §3) can be reinterpreted as translating the symbolic dynamics viewpoint into the world of $\mathbb{R}$-trees, it seems useful to highlight this transition in the symbolic language, before embroiling it with the topology of $\mathbb{R}$-trees:

Remark 2.4. As before, we fix a basis $\mathcal{A}$ of F_N, and denote an element X of the boundary $\partial F_N = \partial F(\mathcal{A})$ by the corresponding infinite reduced word in $\mathcal{A}^{\pm 1}$. We denote by X_n its prefix of length n.

We consider the *unit cylinder* $C^2_{\mathcal{A}}$ in $\partial^2 F_N$:

$$C^2_{\mathcal{A}} = \{(X,Y) \in \partial^2 F_N \mid X_1 \neq Y_1\}$$

Contrary to $\partial^2 F_N$, the unit cylinder $C^2_{\mathcal{A}}$ is a compact set (in fact, a Cantor set). The unit cylinder $C^2_{\mathcal{A}}$ has the property that the canonical map $\rho_{\mathcal{A}} : (X, Y) \mapsto X^{-1} \cdot Y$ (see [CHL-I, Remark 4.3]) restricts to an injection on $C^2_{\mathcal{A}}$ with inverse map $Z \mapsto (Z^-, Z^+)$.

In symbolic dynamics, the natural operator on biinfinite sequences is the *shift* map, which in our notation is given by

$$\sigma(X^{-1} \cdot Y) = X^{-1} Y_1 \cdot (Y_1^{-1} Y)\,,$$

i.e. the same symbol sequence as in $X^{-1} \cdot Y$, but with Y_1 as letter of index 0.

On the other hand, there is a system of "partial bijections" on $C^2_{\mathcal{A}}$, given for each $a_i \in \mathcal{A}$ by:

$$a_i : C^2_{\mathcal{A}} \cap {a_i}^{-1} C^2_{\mathcal{A}} \to a_i C^2_{\mathcal{A}} \cap C^2_{\mathcal{A}}$$

A particular feature of this system is that it "commutes" via the map $\rho_{\mathcal{A}}$ with the shift map σ on the set of biinfinite reduced words: More precisely, for all $(X, Y) \in C^2_{\mathcal{A}}$ one has:

$$\rho_{\mathcal{A}}(Y_1^{-1}(X, Y)) = \sigma(\rho_{\mathcal{A}}(X, Y))$$

This transition from group action to the shift (or more precisely, the converse direction), will be explored in §3 in detail, with the additional feature that the topology of compact trees is added on, in the analogous way as interval exchange transformations are a classical tool to interpret certain symbolic dynamical systems topologically.

2.4 The dual lamination $L(T)$

In [CHL-II] a *dual lamination* $L(T)$ for any isometric action of a free group F_N on an $\mathbb{R}$-tree T has been introduced and investigated. If T is very small and has dense orbits, three different definitions of $L(T)$ have been given in [CHL-II] and shown there to be equivalent. However, as in this paper we can not always assume that T has dense orbits, it is most convenient to fix a basis $\mathcal{A}$ of F_N and to give the general definition of $L(T)$ via its *dual laminary language* $\mathcal{L}_{\mathcal{A}}(T)$ (see Definition 4.1 and Remark 4.2 of [CHL-I]), which determines $L(T)$ and vice versa:

$$\begin{aligned} \mathcal{L}_{\mathcal{A}}(T) \;\; = \;\; \{ v \in F(\mathcal{A}) \mid \forall\, \varepsilon > 0 \; \exists\, u, w \in F(\mathcal{A}) \; : \|u \cdot v \cdot w\|_T < \varepsilon, \\ u \cdot v \cdot w \;\text{ reduced and cyclically reduced} \} \end{aligned}$$

Remark 2.5. It follows directly from this definition that $L(T) = L(T^{min})$, where T^{min} denotes the minimal F_N-invariant subtree of T.

2.5 The map $\mathcal{Q}$

Theorem 2.6 ([LL03, LL08])**.** *Let T be an $\mathbb{R}$-tree with a very small action of F_N by isometries that has dense orbits. Then there exists a surjective F_N-equivariant map $\mathcal{Q} : \partial F_N \to \widehat{T}$ which has the following property:*

For any sequence of elements u_n of F_N which converges to $X \in \partial F_N$ and for any point $P \in T$, if the sequence of points $u_n P \in T$ converges (metrically) in $\widehat{T}$ to a point Q, then $\mathcal{Q}(X) = Q$.

Using the properties of a metric topology, we get the following lemma.

Lemma 2.7. *Let T be an $\mathbb{R}$-tree with a very small action of F_N by isometries that has dense orbits. Let K be a compact (with respect to the metric topology) subtree of $\overline{T}$. Let Q be a point in K and w_n a sequence of elements in F_N which converge in $F_N \cup \partial F_N$ to some $X \in \partial F_N$. If for all n one has $w_n^{-1} Q \in K$, then $\mathcal{Q}(X) = Q$.*

Proof. As K is compact, up to passing to a subsequence, we can assume that $w_n^{-1} Q$ converges to a point P in K, that is to say $\lim_{n\to\infty} d(w_n^{-1} Q, P) = 0$. As the action is isometric, we get that $\lim_{n\to\infty} d(Q, w_n P) = 0$, i.e. the $w_n P$ converge to Q. Hence Theorem 2.6 gives the desired conclusion $\mathcal{Q}(X) = Q$. □

It is crucial for the arguments presented in this paper to remember that the map $\mathcal{Q}$ is not continuous with respect to the *metric topology* on $\widehat{T}$, i.e. the topology given by the metric on T. In fact, this has been the reason why in [CHL05] the weaker observers' topology on $\widehat{T}$ has been investigated.

Theorem 2.8 ([CHL05, Remark 2.2 and Proposition 2.3])**.** *Let T be an $\mathbb{R}$-tree with isometric very small action of F_N that has dense orbits. Then the following holds:*

(1) The map $\mathcal{Q}$ defined in Theorem 2.6 is continuous with respect to the observers' topology, i.e. it defines a continuous equivariant surjection

$$\mathcal{Q} : \partial F_N \to \widehat{T}^{obs}.$$

(2) For any point $P \in T$ the map $\mathcal{Q}$ defines the continuous extension to $F_N \cup \partial F_N$ of the map

$$\mathcal{Q}_P : F_N \to \widehat{T}^{obs}, \ w \mapsto wP\,.$$

Though obvious it is worth noting that the last property determines the map $\mathcal{Q}$ uniquely.

2.6 The map $\mathcal{Q}^2$

If the tree T is very small and has dense orbits, the dual lamination $L(T)$ described in §2.4 admits an alternative second definition via the above defined map $\mathcal{Q}$ as algebraic lamination $L^2(T)$ (compare §2.3):

$$L^2(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}$$

It has been proved in [LL03, LL08] that the map $\mathcal{Q}$ is one-to-one on the preimage of the Gromov boundary ∂T of T. Hence the map $\mathcal{Q}$ induces a map $\mathcal{Q}^2$ from $L^2(T)$ to $\overline{T}$, given by:

$$\mathcal{Q}^2((X, Y)) = \mathcal{Q}(X) = \mathcal{Q}(Y)$$

In light of the above discussion the following result seems remarkable. It is also crucial for the definition of the heart of T in the next subsection.

Proposition 2.9 ([CHL-II, Proposition 8.3])**.** *The F_N-equivariant map*

$$\mathcal{Q}^2 : L^2(T) \to \overline{T}$$

is continuous, with respect to the metric topology on $\overline{T}$.

As in [CHL05, §2], we consider the equivalence relation on ∂F_N whose classes are fibers of $\mathcal{Q}$, and we denote by $\partial F_N/L^2(T)$ the quotient set. The quotient topology on $\partial F_N/L^2(T)$ is the finest topology such that the natural projection $\pi : \partial F_N \to \partial F_N/L^2(T)$ is continuous. The map $\mathcal{Q}$ splits over π, thus inducing a map $\varphi : \partial F_N/L^2(T) \to \widehat{T}^{\mathrm{obs}}$ with $\mathcal{Q} = \varphi \circ \pi$.

Theorem 2.10 ([CHL05, Corollary 2.6])**.** *The map*

$$\varphi : \partial F_N/L^2(T) \to \widehat{T}^{obs}$$

is a homeomorphism.

2.7 The limit set and the heart of T

We consider again the *unit cylinder* $C^2_{\mathcal{A}} = \{(X,Y) \in \partial^2 F_N \mid X_1 \neq Y_1\}$ in $\partial^2 F_N$ as defined in Remark 2.4. The following definition is the crucial innovative tool of this paper:

Definition 2.11. The *limit set* of T with respect to the basis $\mathcal{A}$ is the set

$$\Omega_{\mathcal{A}} = \mathcal{Q}^2(C^2_{\mathcal{A}} \cap L^2(T)) \subset \overline{T}\,.$$

The *heart* $K_{\mathcal{A}}$ of T with respect to the basis $\mathcal{A}$ is the convex hull in $\overline{T}$ of the limit set $\Omega_{\mathcal{A}}$.

It is not hard to see that in any $\mathbb{R}$-tree the convex hull of a compact set is again compact. Thus we obtain from Proposition 2.9 and Definition 2.11:

Corollary 2.12. *The limit set $\Omega_{\mathcal{A}}$ is a compact subset of $\overline{T}$. The heart $K_{\mathcal{A}} \subset \overline{T}$ is a compact $\mathbb{R}$-tree.*

Note that, while $L^2(T)$ does not depend on the choice of the basis $\mathcal{A}$, the unit cylinder $C^2_{\mathcal{A}}$ and thus the limit set and the heart of T do crucially depend on the choice of $\mathcal{A}$.

3 Systems of isometries on compact $\mathbb{R}$-trees

In this section we review the basic construction that associates an $\mathbb{R}$-tree to a system of isometries. This goes back to the seminal papers of D. Gaboriau, G. Levitt, and F. Paulin [GLP94] and M. Bestvina and M. Feighn [BF95], and before them to the study of surface trees and the work of J. Morgan and P. Shalen [MS91], R. Skora [Sko96], and of course to the fundamental work of E. Rips.

3.1 Definitions

Definition 3.1. (a) Let K be a compact $\mathbb{R}$-tree. A *partial isometry* of K is an isometry between two closed subtrees of K. It is said to be *non-empty* if its domain is non-empty.

(b) A *system of isometries* $\mathcal{K} = (K, \mathcal{A})$ consists of a compact $\mathbb{R}$-tree K and a finite set $\mathcal{A}$ of non-empty partial isometries of K. This defines a *pseudo-group of partial isometries* of K by admitting inverses and composition.

We note that in the literature mentioned above it is usually required that K is a finite tree, i.e. K is a metric realisation of a finite simplicial tree, or, equivalently, K is the convex hull of finitely many points. The novelty here is that we only require K to be compact. Recall that a compact $\mathbb{R}$-tree K may well have infinitely many branch points, possibly with infinite valence, and that K may well contain finite trees of unbounded volume (but of course K has finite diameter). In the context of this paper, however, all trees have a countable number of branch points, which makes compact trees slightly more tractable.

Any element of the free group F_N over the basis $\mathcal{A}$, given as reduced word $w = z_1 \dots z_n \in F(\mathcal{A})$, defines a (possibly empty) partial isometry, also denoted by w, which is defined as the composition of partial isometries $z_1 \circ z_2 \circ \dots \circ z_n$. We write this *pseudo-action* of $F(\mathcal{A})$ on K on the right, i.e.

$$x(u \circ v) = (xu)v$$

for all $x \in K$ and $u, v \in F(\mathcal{A})$. For any points $x, y \in K$ and any $w \in F(\mathcal{A})$ we obtain

$$xw = y$$

if and only if x is in the domain $\mathrm{dom}(w)$ of w and is sent by w to y.

A reduced word $w \in F(\mathcal{A})$ is called *admissible* if it is non-empty as a partial isometry of K.

3.2 The $\mathbb{R}$-tree associated to a system of isometries

A system of isometries $\mathcal{K} = (K, \mathcal{A})$ defines an $\mathbb{R}$-tree $T_{\mathcal{K}}$, provided with an action of the free group $F_N = F(\mathcal{A})$ by isometries. The construction is the same as in the case where K is a finite tree and will be recalled now.

As in [GL95] the tree $T_{\mathcal{K}}$ can be described using a foliated band-complex, but for non-finite K one would not get a CW-complex. We use the following equivalent construction in combinatorial terms.

The tree $T_{\mathcal{K}}$ is obtained by gluing countably many copies of K along the partial isometries, one for each element of F_N. On the topological space $F_N \times K$ these identifications are made formal by defining

$$T_{\mathcal{K}} = F_N \times K / \sim$$

where the equivalence relation $\sim$ is defined by:

$$(u, x) \sim (v, y) \iff x(u^{-1}v) = y$$

The free group F_N acts on $T_{\mathcal{K}}$, from the left: this action is simply given by left-multiplication on the first coordinate of each pair $(u, x) \in F_N \times K$:

$$w(u, x) = (wu, x)$$

for all $u, w \in F_N, x \in K$.

Since F_N is free over $\mathcal{A}$, each copy $\{u\} \times K$ of K embeds canonically into $T_{\mathcal{K}}$. Thus we can identify K with the image of $\{1\} \times K$ in $T_{\mathcal{K}}$, so that every $\{u\} \times K$ maps bijectively onto uK. Using these bijections, the metric on K defines canonically a pseudo-metric on $T_{\mathcal{K}}$. Again, by the freeness of F_N over $\mathcal{A}$, this pseudo-metric is a metric. The arguments given in the proof of Theorem I.1. of [GL95] extend directly from the case of finite K to compact K, to show:

Theorem 3.2. *Given a system of isometries $\mathcal{K} = (K, \mathcal{A})$ on a compact $\mathbb{R}$-tree K, there exists a unique $\mathbb{R}$-tree $T_{\mathcal{K}}$, provided with a left-action of $F(\mathcal{A})$ by isometries, which satisfies:*

(1) $T_{\mathcal{K}}$ contains K (as an isometrically embedded subtree).

(2) If $x \in K$ is in the domain of $a \in \mathcal{A}$, then $a^{-1}x = xa$.

(3) Every orbit of the $F(\mathcal{A})$-action on $T_{\mathcal{K}}$ meets K. Indeed, every segment of $T_{\mathcal{K}}$ is contained in a finite union of translates w_iK, for suitable $w_i \in F(\mathcal{A})$.

(4) If T is another $\mathbb{R}$-tree with an action of $F(\mathcal{A})$ by isometries satisfying (1) and (2), then there exists a unique $F(\mathcal{A})$-equivariant morphism $j : T_{\mathcal{K}} \to T$ such that $j(x) = x$ for all $x \in K$. □

3.3 Systems of isometries induced by an F_N-action on an $\mathbb{R}$-tree

Frequent and important examples of systems of isometries occur in the following context:

Let T be any $\mathbb{R}$-tree with an $F(\mathcal{A})$-action by isometries. Then any compact subtree $K \subset T$, which is sufficiently large so that it intersects for any $a_i \in \mathcal{A}$ the translate a_iK, defines canonically a system of isometries given by:

$$\begin{array}{rrcl} a_i : & a_iK \cap K & \to & K \cap a_i^{-1}K \\ & x & \mapsto & xa_i = a_i^{-1}x \end{array}$$

Since K embeds into T, Theorem 3.2 gives a map

$$j : T_{\mathcal{K}} \to T.$$

The map j fails in general to be injective. A classical technique for the study of an action on an $\mathbb{R}$-tree T is to view $T_{\mathcal{K}}$ as an approximation of T, and to consider a sequence of increasing K. As K increases to exhaust T, the convergence of the sequence of $T_{\mathcal{K}}$ to T is well understood. Moreover, if K is a finite subtree of T, then $T_{\mathcal{K}}$ is called geometric and the full strength of the Rips machine can be used to study it

In this article, we propose a new approach to study T, namely we prove that there exists a compact subtree K of $\overline{T}$ such that j is an isometry. This gives the possibility to extend the results proved for geometric trees (i.e. when K is finite) to the case where K is only assumed to be compact.

3.4 Basic lemmas

We now present some basic lemmas about the action on $T_{\mathcal{K}}$, for admissible and non-admissible words in the given system of isometries. We first observe:

Remark 3.3. (a) Let K and K' be two closed disjoint subtrees of T. Then there exists a unique segment $[x, x']$ which *joins K to K'*, i.e. one has $K \cap [x, x'] = \{x\}$ and $K' \cap [x, x'] = \{x'\}$. For any further points $y \in K, y' \in K'$ the segment $[y, y']$ contains both segments $[x, y']$ and $[x', y]$, and both contain $[x, x']$.

(b) As a shorthand, we use in the situation given above the following notation:

$$[K, K'] := [x, x'], \;\; [y, K'] := [y, x'], \;\; [K, y'] := [x, y']$$

(c) If $y \in K$, then we set $[y, K] = [K, y] = \{y\}$, i.e. the segment of length 0 with y as initial and terminal point.

The following is a specification of statement (3) of Theorem 3.2:

Lemma 3.4. *For any non-admissible word $w \in F(\mathcal{A})$ one has*

$$[K, wK] \;\; \subset \;\; \bigcup_{i=0}^{|w|} w_i K \,,$$

where w_i is the prefix of w with length $|w_i| = i$.

Proof. It suffices to show that for the reduced word $w = z_1 \ldots z_n$ the union $\bigcup\limits_{i=0}^{n} w_i K$ is connected. This follows directly from the fact that for all $i = 1, \ldots, n$ the union $w_{i-1}K \cup w_i K = w_{i-1}(K \cup w_{i-1}^{-1} w_i K)$ is connected, since $w_{i-1}^{-1} w_i = z_i \in \mathcal{A}^{\pm 1}$, and all partial isometries from $\mathcal{A}$ are assumed to be non-empty. $\square$

Lemma 3.5. *Let $\mathcal{K} = (K, \mathcal{A}), T_{\mathcal{K}}$ and $F(\mathcal{A})$ be as above.*

(1) For all $w \in F(\mathcal{A})$ one has

$$dom(w) = K \cap wK \,.$$

(2) A word $w \in F(\mathcal{A})$ is admissible if and only if $K \cap wK \neq \emptyset$.

(3) If $x \in dom(w)$, then

$$w^{-1}x = xw \,.$$

Proof. Let $w \in F(\mathcal{A})$ and $x \in T_{\mathcal{K}}$. If $x \in \mathrm{dom}(w) \subset K$, then the definition of $T_{\mathcal{K}}$ gives $(1, x) \sim (w, xw)$, or equivalently (compare Theorem 3.2)

$$w^{-1}x = xw\,.$$

Therefore x is contained in both K and wK. This shows:

$$\mathrm{dom}(w) \subset K \cap wK$$

Conversely, let x be in $K \cap wK$. Then $(1, x) \sim (w, y)$ for some point $y \in K$, and by definition of $\sim$ the point x lies in the domain of w, with $xw = y$. Thus w is admissible, and

$$K \cap wK \subset \mathrm{dom}(w)\,.$$

□

Lemma 3.6. *For all $w \in F(\mathcal{A})$ the following holds, where w_k denotes the prefix of w of length k:*

(1) $\quad dom(w) \subset dom(w_k) \quad$ *for all $k \leq |w|$.*

(2) $\quad dom(w) = \bigcap_{k=0}^{|w|} w_k K$

Proof. Assertion (1) follows directly from the definition of $\mathrm{dom}(w)$. Assertion (2) follows from assertion (1) and Lemma 3.5 (1). □

Remark 3.7. We would like to emphasize that it is important to keep the $F(\mathcal{A})$-action on $T_{\mathcal{K}}$ apart from the $F(\mathcal{A})$-pseudo-action on K. This is the reason why we define the action on $T_{\mathcal{K}}$ from the left, whereas we define the pseudo-action by partial isometries on K from the right.

This setting is also convenient to keep track of the two actions: a point $x \in K$ lies in the domain of the partial isometry associated to $w \in F(\mathcal{A})$ if and only if x is contained in wK (Lemma 3.5 (1)). More to the point, the sequence of partial isometries given by the word $w = z_1 \dots z_n$ defines points $xz_1 \dots z_i$ which lie all inside of K if and only if the sequence of isometries of T given by the prefixes of w moves K within T in such a way that x is contained in each of the translates $z_1 \dots z_i K$ (see Lemma 3.6 (2)).

Lemma 3.8. *(a) For any non-admissible word $w \in F(\mathcal{A})$ and any disjoint closed subtrees K and wK, the arc $[K, wK]$ intersects all $w_i K$, where w_i is a prefix of w.*

(b) For any point $Q \in K$ and any (possibly admissible) word $w \in F(\mathcal{A})$, the arc $[Q, wK]$ intersects all $w_i K$.

Proof. (a) We prove part (a) by induction on the length of w.

Let u be the longest admissible prefix of w. Thus $u \neq 1$, as all partial isometries in $\mathcal{A}^{\pm 1}$ are non-empty. Hence we can assume by induction that $u^{-1}w$ is either admissible or satisfies the property stated in part (a).

Let a be the next letter of w after the prefix u. We write w as reduced product $w = u \cdot a \cdot v$. According to Lemma 3.5 (2) one has:

(i) $uK \cap K = \mathrm{dom}(u) \neq \emptyset$

(ii) $uK \cap uaK = u\,\mathrm{dom}(a) \neq \emptyset$, and

(iii) $K \cap ua\,K = \emptyset$

By (iii) there is a non-trivial segment $\beta = [K, uaK] \subset T_{\mathcal{K}}$ that intersects K and uaK only in its endpoints. By (i) and (ii) the segment β is contained in the subtree uK: there are points $x, y \in K$ such that $\beta = [ux, uy]$. Since ux belongs to $K \cap uK = \mathrm{dom}(u)$, it follows from Lemma 3.6 (2) that ux also belongs to every $u'K$, for any prefix u' of u.

Moreover, for any prefix v' of v one has, by Lemma 3.5 (1) and Lemma 3.6 (1):

$$uav'K \cap uK \;=\; u\,\mathrm{dom}(av') \;\subset\; u\,\mathrm{dom}(a) \;=\; uaK \cap uK$$

From this we deduce that

$$\begin{array}{rcll} uav'K \cap [ux, uy] & \subset & uav'K \cap [ux, uy] \cap uK & \\ & \subset & [ux, uy] \cap uaK \cap uK & \\ & \subset & [ux, uy] \cap uaK & = \;\; \{uy\}\,. \end{array}$$

Since the segment $\alpha = [K, wK]$ is by Lemma 3.4 contained in the union

$$\bigcup_{i=0}^{|w|} w_i K$$

it follows from the above derived inclusion $uav'K \cap [ux, uy] \subset \{uy\}$ that α is the union of $\beta = [ux, uy]$ and of the segment $\gamma = [uy, wK]$, with $\beta \cap \gamma = \{uy\}$.

If av is admissible, then the endpoint of γ is contained in the intersection of all $uav'K$, by Lemma 3.6 (2). If av is non-admisible, we apply the induction hypothesis to $u^{-1}w = av$ and obtain that every $av'K$ meets the arc $\gamma' = [K, avK]$. But $u\gamma'$ is a subarc of γ, so that the arc $[ux, uy] \cup \gamma$ meets infact all w_iK, as claimed.

(b) In case that w is non-admissible, there is a largest index i such that $K \cap w_iK \neq \emptyset$. We can now apply statement (a) to $w_i^{-1}K$ and $w_i^{-1}w$ to get the desired conclusion.

If w is admissible, then $\mathrm{dom}(w) = K \cap wK$ (by Lemma 3.5 (1)). Hence the arc $[Q, wK]$ is contained in K, and by Lemma 3.6 (2) its endpoint is contained in any w_iK. □

Lemma 3.9. *Let $w, w' \in F(\mathcal{A})$ with maximal common prefix $u \in F(\mathcal{A})$. Then for any triplet of points $Q \in K$, $R \in wK$ and $R' \in w'K$ the arcs $[Q, R]$ and $[Q, R']$ intersect in an arc $[Q, P]$ with endpoint $P \in uK$.*

Proof. Let $[Q, Q_1]$ the arc which joins K to uK. It follows directly from Lemma 3.8 (b) that Q_1 lies on both $[Q, R]$ and $[Q, R']$. Similarly, let $[R, R_1]$ and $[R', R'_1]$ be the arcs that join R to uK and R' to uK respectively. After applying w^{-1} or ${w'}^{-1}$ we obtain in the same way that R_1 lies on both $[Q, R]$ and $[R, R']$, and that R'_1 lies on both $[Q, R']$ and $[R, R']$. Hence the geodesic triangle in $T_{\mathcal{K}}$ with endpoints Q, R, R' contains the geodesic triangle with endpoints Q_1, R_1 and R'_1, and the center of the latter is equal to the center P of the former. But Q_1, R_1 and R'_1 are all three contained in uK, so that P is contained in uK. □

In the following statement and its proof we use the standard terminology for group elements acting on trees, as recalled in §2.1 above.

Proposition 3.10. *Let $w \in F(\mathcal{A})$ is any cyclically reduced word. If the action of w on $T_{\mathcal{K}}$ is hyperbolic, then the axis of w intersects K. If the action of w on $T_{\mathcal{K}}$ is elliptic, then w has a fixed point in K.*

Proof. If w is not admissible, let $[x, wy]$ be the segment that joins K to wK: these two translates are disjoint by Lemma 3.5 (2). As w acts as an isometry, $[wx, w^2y]$ is the segment that joins wK to w^2K. Moreover, since w is assumed to be cyclically reduced, the segment that joins K to w^2K intersects wK, by Lemma 3.8.

Any two consecutive segments among $[x, wy]$, $[wy, wx]$, $[wx, w^2y]$ and $[w^2y, w^2x]$ have precisely one point in common, by Remark 3.3, and hence their union is a segment. This proves that wx belongs to $[x, w^2x]$, and that x is contained in the axis of w.

If w is admissible, then either there exists $n \geq 0$ such that w^n is not admissible, in which case we can fall back on the above treated case, as w and w^n have the same axis. Otherwise, for arbitrary large n there exists a point $x \in K$ such that $w^n x \in K$, by Lemma 3.5 (2). But K is compact and hence has finite diameter. This implies that the action of w on T is not hyperbolic, and hence it is elliptic: w fixes a point of T. Some such fixed point lies on $[x, wx]$ (namely its center), and hence in the subtree K. □

3.5 Admissible laminations

In this subsection we use the concepts of *algebraic lamination*, *symbolic lamination* and *laminary language* as defined in [CHL-I], and the equivalence between these three points of view shown there. The definitions and the notation have been reviewed in §2.3 above.

For any system of isometries $\mathcal{K} = (K, \mathcal{A})$ denote by $\mathrm{Adm}(\mathcal{K}) \subset F(\mathcal{A})$ the set of admissible words. The set $\mathrm{Adm}(\mathcal{K})$ is stable with respect to passage to subwords, but it is not *laminary* (see [CHL-I, Definition 5.2]): not every admissible word w is necessarily equal, for all $k \in \mathbb{N}$, to the word $v\dagger_k$ obtained from some larger $v \in \mathrm{Adm}(\mathcal{K})$ by "chopping off" the two boundary subwords of length k. As does any infinite subset of $F(\mathcal{A})$, the set $\mathrm{Adm}(\mathcal{K})$ generates a laminary language, denoted $\mathcal{L}_{\mathrm{adm}}(\mathcal{K})$, which is the largest laminary language made of admissible words:

$$\mathcal{L}_{\mathrm{adm}}(\mathcal{K}) = \{w \in F(\mathcal{A}) \mid \forall k \in \mathbb{N}\ \exists v \in \mathrm{Adm}(\mathcal{K}) : w = v\dagger_k\}$$

Clearly one has $\mathcal{L}_{\mathrm{adm}}(\mathcal{K}) \subset \mathrm{Adm}(\mathcal{K})$, but the converse is in general false.

As explained in §2.3, any laminary language determines an algebraic lamination (i.e. a closed F_N-invariant and flip-invarinat subset of $\partial^2 F_N$), and conversely. The algebraic lamination determined by $\mathcal{L}_{\mathrm{adm}}(\mathcal{K})$ is called *admissible lamination*, and denoted by $L_{\mathrm{adm}}(\mathcal{K})$.

An infinite word $X \in \partial F(\mathcal{A})$ is *admissible* if all of its prefixes X_n are admissible. The set of admissible infinite words is denoted by $L^1_{\mathrm{adm}}(\mathcal{K})$. It is a closed subset of $\partial F(\mathcal{A})$ but it is not invariant under the action of $F(\mathcal{A})$.

For any infinite admissible X the *domain* $\mathrm{dom}(X)$ of X is defined to be the intersection of all domains $\mathrm{dom}(X_n)$. Since K is compact, one has

$$\mathrm{dom}(X) \neq \emptyset$$

for all $X \in L^1_{\mathrm{adm}}(\mathcal{K})$.

A biinfinite indexed reduced word $Z = \dots z_{-1} z_0 z_1 \dots$, with $z_i \in \mathcal{A}^{\pm}$, is called *admissible*, if its two halves $Z^+ = z_1 z_2 \dots$ and $Z^- = z_0^{-1} z_{-1}^{-1} \dots$ are admissible, and if the intersection of the domains of Z^+ and Z^- is non-empty. The *domain* of Z is defined to be this intersection:

$$\mathrm{dom}(Z) = \mathrm{dom}(Z^+) \cap \mathrm{dom}(Z^-)$$

We observe that Z is admissible if and only if all its subwords are admissible.

The set of biinfinite admissible words is called the *admissible symbolic lamination* of the system of isometries $\mathcal{K} = (K, \mathcal{A})$.

We use now the notion of the dual lamination of an $\mathbb{R}$-tree with isometric F_N-action as introduced in [CHL-II] and reviewed above in §2.4.

Proposition 3.11. *For any system of isometries $\mathcal{K}$ one has*

$$L(T_{\mathcal{K}}) \subseteq L_{adm}(\mathcal{K}).$$

Proof. Let $u \in F(\mathcal{A})$ be a non-admissible word, and let $\varepsilon = d(K, uK)$. By Lemma 3.5 (2) one has $\varepsilon > 0$. Let w be a cyclically reduced word that contains u as a subword: we write $w = u_1 \cdot u \cdot u_2$ as a reduced product. By Proposition 3.10, the axis of w passes through K. But if x is any point in K, the segment $[x, wx]$ contains the segment that joins the disjoint subtrees $u_1 K$ and $u_1 u K$, by Lemma 3.8, and hence the translation length of w, which is realized on its axis, is bigger than ε. This proves that u is not in $\mathcal{L}(T_{\mathcal{K}})$ (see §2.4) .

As the laminary language of $\mathcal{L}_{\mathrm{adm}}(\mathcal{K})$ is the largest laminary language made of admissible words, this concludes the proof. $\square$

4 The map $\mathcal{Q}_{\mathcal{K}}$ for a system of isometries

In this section we define the map $\mathcal{Q}_{\mathcal{K}}$ and we prove that it is the equivalent of the map $\mathcal{Q}$ from §2.5, for systems of isometries $\mathcal{K}$. For this definition we distinguish two cases: If $X \in \partial F(\mathcal{A})$ is not eventually admissible we define $\mathcal{Q}_{\mathcal{K}}(X)$ in §4.1. If X is eventually admissible, the definition of $\mathcal{Q}_{\mathcal{K}}(X)$ is given in §4.3, and in this case we need the hypothesis that the system of isometries has independent generators. Both cases are collected together in §4.4 to obtain a continuous equivariant map $\mathcal{Q}_{\mathcal{K}}$.

4.1 The map $\mathcal{Q}_{\mathcal{K}}$ for non-eventually admissible words

As in §3, let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K, and let $T_{\mathcal{K}}$ be the associated $\mathbb{R}$-tree, provided with an action of the free group $F(\mathcal{A})$ by isometries. Let $X \in \partial F(\mathcal{A})$ be an infinite reduced word and denote as before by X_i the prefix of X of length $i \geq 0$.

Definition 4.1. An infinite word $X \in \partial F(\mathcal{A})$ is *eventually admissible* if there exists an index i such that the suffix ${X_i}^{-1}X$ of X is admissible.

Note that an infinite word $X \in \partial F(\mathcal{A})$ is not eventually admissible if for every index $i \geq 0$ there is an index $j > i$ such that the subword $X_{[i+1,j]} = X_i^{-1}X_j$ of X between the indices $i+1$ and j is not admissible.

Let $X \in \partial F(\mathcal{A})$ be not eventually admissible, and let $i_0 > 0$ be such that the prefix X_{i_0} of X of length i_0 is not admissible. Then for any $i \geq i_0$, the prefix X_i is not admissible, and thus, by Lemma 3.5, K and X_iK are disjoint. By Lemmas 3.8 and 3.9, for any $j \geq i \geq i_0$ the segment $[K, X_iK]$ and $[K, X_jK]$ are nested and have the same initial point $Q \in K$. Let Q_i be the terminal point of $[K, X_iK]$:

$$[Q, Q_i] = [K, X_iK]$$

The sequence of Q_i converges in $\widehat{T}_{\mathcal{K}}$ with respect to both the metric and the observers' topology. Moreover, the two limits are the same.

Definition 4.2. For any $X \in \partial F(\mathcal{A})$ which is not eventually admissible, we define:

$$\mathcal{Q}_{\mathcal{K}}(X) = \lim_{i\to\infty} Q_i$$

Proposition 4.3. *Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K. Let $X \in \partial F(\mathcal{A})$ be not eventually admissible.*

Let $w_n \in F(\mathcal{A})$ be a sequence of words which converge in $F(\mathcal{A}) \cup \partial F(\mathcal{A})$ to X, and let $P_n \in w_nK$. Then the sequence of points P_n converges in $\widehat{T}^{obs}_{\mathcal{K}}$ to $\mathcal{Q}_{\mathcal{K}}(X)$, and $\mathcal{Q}_{\mathcal{K}}(X)$ belongs to $\widehat{T}^{obs}_{\mathcal{K}} \smallsetminus T_{\mathcal{K}}$.

Proof. We use the above notations. For every index $i \geq 0$, let $[Q_i, R_i]$ be the intersection of $[Q, \mathcal{Q}_{\mathcal{K}}(X)]$ with X_iK. Hence for $i \geq i_0$ the point Q_i is, as before, the terminal point of the segment $[K, X_iK]$. The segments $[Q, Q_i]$ are increasingly nested, the segments $[R_i, \mathcal{Q}_{\mathcal{K}}(X)]$ are decreasingly nested, Q_i is a point of $[Q, R_i]$ and R_i is a point of $[Q_i, \mathcal{Q}_{\mathcal{K}}(X)]$.

As X is not eventually admissible, for every index $i \geq 0$ there is an index $j > i$ such that the subword $X_{[i+1,j]}$ of X between the indices $i+1$ and j is not admissible. By Lemma 3.5 the segments $[Q_i, R_i]$ and $[Q_j, R_j]$ are disjoint.

For any n, let $i(n)$ be the length of the maximal common prefix of w_n and X. By Lemma 3.9, the maximal common segment $[Q, P'_n]$ of $[Q, P_n]$ and $[Q, \mathcal{Q}_{\mathcal{K}}(X)]$

has its terminal point P'_n in $[Q_{i(n)}, R_{i(n)}]$. As X is not eventually admissible, for m big enough the subword $X_{[i(n)+1,i(m)]}$ of X between the indices $i(n)+1$ and $i(m)$ is not admissible and the segments $[Q_{i(n)}, R_{i(n)}]$ and $[Q_{i(m)}, R_{i(m)}]$ are disjoint. Therefore the maximal common segment of $[Q, P_n]$ and $[Q, P_m]$ is also the maximal common segment of $[Q, P_n]$ and $[Q, \mathcal{Q}_{\mathcal{K}}(X)]$, and hence it is equal to $[Q, P'_n]$.

The points P'_n converge to $\mathcal{Q}_{\mathcal{K}}(X)$, as any sequence of points in $[Q_{i(n)}, R_{i(n)}]$ does, and this proves that

$$\liminf{}_Q P_n = \mathcal{Q}_{\mathcal{K}}(X).$$

By Lemma 2.2 any subsequence of P_n, which converges in $\widehat{T}^{\text{obs}}_{\mathcal{K}}$, necessarily converges to $\mathcal{Q}_{\mathcal{K}}(X)$. Hence by compactness of $\widehat{T}^{\text{obs}}_{\mathcal{K}}$, the sequence of all of the points P_n converges to $\mathcal{Q}_{\mathcal{K}}(X)$ with respect to the observers' topology.

If P is a point in uK for some u in F_N, then the maximal common segment $[Q, P']$ of $[Q, P]$ and $[Q, \mathcal{Q}_{\mathcal{K}}(X)]$ has its endpoint P' in $[Q_i, R_i]$, where X_i is the maximal common prefix of u and X. Thus $P' \neq \mathcal{Q}_{\mathcal{K}}(X)$, and hence $\mathcal{Q}_{\mathcal{K}}(X)$ is not contained in $T_{\mathcal{K}}$. □

4.2 Independent generators

The following concept is due to Gaboriau [Gab97], in the case of finite K, and we extend it here to the compact case.

Definition 4.4. Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K. Then $\mathcal{K}$ is said to have *independent generators* if, for any infinite admissible word $X \in \partial F(\mathcal{A})$, the non-empty domain of X consists of exactly one point.

The same arguments as in [Gab97] show the following equivalences. However, they will not be used in the sequel.

Remark 4.5. Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K. The following are equivalent:

(1) $\mathcal{K}$ has independent generators.

(2) Every non-trivial admissible word fixes at most one point of K.

(3) The action of $F(\mathcal{A})$ on the associated tree $T_{\mathcal{K}}$ has trivial arc stabilizers.

Note that Gaboriau [Gab97] used originally property (2) as definition, but in our context this seems less natural.

4.3 The map $\mathcal{Q}_{\mathcal{K}}$ for eventually admissible words

Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K. Consider the set $L^1_{\text{adm}}(\mathcal{K}) \subset \partial F(\mathcal{A})$ of infinite admissible words as defined in §3.5.

Definition 4.6. Let $\mathcal{K}$ be a system of isometries which has independent generators. Then for any infinite admissible word $X \in L^1_{\text{adm}}(\mathcal{K})$ there exists exactly one element of K in the domain of X, which will be called $\mathcal{Q}_{\mathcal{K}}(X)$.

Lemma 4.7. *Identify K with the image of $\{1\} \times K$ in $\widehat{T}_{\mathcal{K}}$ as in §3, and let $X \in L^1_{adm}(\mathcal{K})$.*

(1) Denoting as before by X_i the prefix of X of length $i \geq 1$, we obtain:

$$\{\mathcal{Q}_{\mathcal{K}}(X)\} = \bigcap_{i \geq 1} X_i K$$

(2) For every $i \geq 1$ we have:

$$\mathcal{Q}_{\mathcal{K}}(X_i^{-1} X) = X_i^{-1} \mathcal{Q}_{\mathcal{K}}(X)$$

Proof. Assertion (1) follows directly from Lemma 3.6 (2) and the above definition of the map $\mathcal{Q}_{\mathcal{K}}$. Assertion (2) follows directly from (1). □

Recall from Definition 4.1 that an infinite words $X \in \partial F(\mathcal{A})$ is eventually admissible if it has a prefix X_i such that the infinite remainder $X'_i = X_i^{-1} X$ is admissible. We observe that for all integers $j \geq i$ the word $X_i^{-1} X_j$ is admissible, so that Lemma 4.7 (2) gives:

$$X_i \mathcal{Q}_{\mathcal{K}}(X'_i) = X_i \mathcal{Q}_{\mathcal{K}}(X_i^{-1} X_j X'_j) = X_i (X_i^{-1} X_j) \mathcal{Q}_{\mathcal{K}}(X'_j) = X_j \mathcal{Q}_{\mathcal{K}}(X'_j)$$

Hence the following definition does not depend on the choice of the index i.

Definition 4.8. For any eventually admissible word $X \in \partial F(\mathcal{A})$ we define

$$\mathcal{Q}_{\mathcal{K}}(X) = X_i \mathcal{Q}_{\mathcal{K}}(X'_i).$$

We note that for any element $u \in F(\mathcal{A})$ and any eventually admissible word $X \in \partial F(\mathcal{A})$ one has:

$$\mathcal{Q}_{\mathcal{K}}(uX) = u \mathcal{Q}_{\mathcal{K}}(X)$$

Proposition 4.9. *Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K with indenpendent generators. Let $X \in \partial F(\mathcal{A})$ be an eventually admissible word.*

For any element P in $T_{\mathcal{K}}$, and any sequence w_n of elements of F_N that converge to X, the sequence of points $w_n P$ converges to $\mathcal{Q}_{\mathcal{K}}(X)$, with respect to the observers' topology on $T_{\mathcal{K}}$.

Proof. Up to multiplying by the inverse of a prefix we can assume that X is admissible and $\mathcal{Q}_{\mathcal{K}}(X) \in K$. By compactness of $\widehat{T}^{\text{obs}}_{\mathcal{K}}$ we can assume that $w_n P$ converges to some point Q_∞. By contradiction assume that $Q_\infty \neq \mathcal{Q}_{\mathcal{K}}(X)$, and let M be a point in the open interval $(Q_\infty, \mathcal{Q}_{\mathcal{K}}(X))$. From Lemma 2.2 we deduce

$$Q_\infty = \liminf{}_{\mathcal{Q}_{\mathcal{K}}(X)} w_n P.$$

Thus, for n and m big enough, the maximal common segment $[\mathcal{Q}_{\mathcal{K}}(X), P_{m,n}]$ of the segments $[\mathcal{Q}_{\mathcal{K}}(X), w_n P]$ and $[\mathcal{Q}_{\mathcal{K}}(X), w_m P]$ contains M. As w_n converges to X, for n fixed and for m sufficiently large, the maximal common prefix of w_n and w_m is a prefix X_i of X. By Lemma 3.9, $P_{m,n}$ is contained in $X_i K$. By Lemma 4.7, $\mathcal{Q}_{\mathcal{K}}(X)$ is also contained $X_i K$, and hence, so is M. As m and n grow larger, the index i goes to infinity (since $w_n \to X$), which proves that M is contained in the intersection of all the $X_i K$. Since we assumed $M \neq \mathcal{Q}_{\mathcal{K}}(X)$, this contradicts the independent generators' hypothesis. □

4.4 Continuity of the map $\mathcal{Q}_{\mathcal{K}}$

As any element of $\partial F(\mathcal{A})$ is either eventually admissible or not, from Definitions 4.2 and 4.8 we collect a map $\mathcal{Q}_{\mathcal{K}}$.

Corollary 4.10. *Let $\mathcal{K} = (K, \mathcal{A})$ be a system of isometries on a compact $\mathbb{R}$-tree K with independent generators. The map $\mathcal{Q}_{\mathcal{K}} : \partial F_N \to \widehat{T}^{obs}_{\mathcal{K}}$ is equivariant and continuous.*

For any point P in $T_{\mathcal{K}}$, the map $\mathcal{Q}_{\mathcal{K}}$ defines the continuous extension to $F_N \cup \partial F_N$ of the map

$$\begin{array}{rccc} Q_P : & F_N & \to & \widehat{T}^{obs}_{\mathcal{K}} \\ & w & \mapsto & wP \end{array}$$

Proof. Equivariance and continuity of $\mathcal{Q}_{\mathcal{K}}$ follow from the second part of the statement, which is proved in Propositions 4.3 and 4.9. □

5 Proof of the Main Theorem

Throughout this section let T be an $\mathbb{R}$-tree provided with a minimal, very small action of F_N by isometries which has dense orbits. Hence we obtain from Theorem 2.6 an equivariant and continuous map $\mathcal{Q}$, which we denote here by $\mathcal{Q}_T : \partial F_N \to \widehat{T}^{\mathrm{obs}}$.

Let $\mathcal{A}$ be a basis of F_N, and let K be a compact subtree of $\overline{T}$. Let $\mathcal{K} = (K, \mathcal{A})$ be the induced system of isometries $a_i : K \cap a_i K \to a_i^{-1} K \cap K$, $x \mapsto x a_i = a_i^{-1} x$, as discussed in §3.3. We assume that K is chosen large enough so that for each $a_i \in \mathcal{A}$ the intersection $K \cap a_i K$ and hence the partial isometry $a_i \in \mathcal{A}$ is non-empty. As a consequence (see §3), there exists an $\mathbb{R}$-tree $T_{\mathcal{K}}$ with isometric action by F_N, and by Theorem 3.2 there exists a unique continuous F_N-equivariant map

$$j : T_{\mathcal{K}} \to \overline{T}$$

which induces the identity map $T_K \supset K \xrightarrow{j} K \subset \overline{T}$.

Lemma 5.1. *The system of isometries $\mathcal{K} = (K, \mathcal{A})$ has independent generators.*

Proof. Let Q be a point in the domain of an infinite admissible word X, compare §3.5. Then for any prefix X_n of X, the point $QX_n = {X_n}^{-1} Q$ is also contained in K (recall that we write the action of $F(\mathcal{A})$ on $T_\mathcal{K}$ on the left, and the pseudo-action of partial isometries of $\mathcal{K}$ on the right).

By Theorem 3.2, j restricts to an isometry between $K \subset T_\mathcal{K}$ and $K \subset T$. Therefore, for any $n \geq 0$, ${X_n}^{-1} j(Q)$ lies in $K \subset T$. By Lemma 2.7, we get $\mathcal{Q}_T(X) = j(Q)$.

This proves that the domain of X consists of at most the point $j^{-1}(\mathcal{Q}_T(X))$. Hence $\mathcal{K}$ has independent generators. $\square$

As a consequence of Lemma 5.1, we can apply Corollary 4.10 to obtain an equivariant and continuous map $\mathcal{Q}_\mathcal{K} : \partial F_N \to \widehat{T}_\mathcal{K}^{\text{obs}}$.

Lemma 5.2. *For any $X \in \partial F_N$ such that $\mathcal{Q}_\mathcal{K}(X)$ is contained in $T_\mathcal{K}$, one has*

$$j(\mathcal{Q}_\mathcal{K}(X)) = \mathcal{Q}_T(X).$$

Proof. By Proposition 4.3, X is eventually admissible and by equivariance of $\mathcal{Q}_\mathcal{K}$, $\mathcal{Q}_T$ and j, we can assume that X is admissible and that $\mathcal{Q}_\mathcal{K}(X)$ is in K. By Definition 4.6, for any $i \geq 0$, $\mathcal{Q}_\mathcal{K}(X) \cdot X_i = {X_i}^{-1} \mathcal{Q}_\mathcal{K}(X)$ lies in K.

By Theorem 3.2, j restricts to an isometry between $K \subset T_\mathcal{K}$ and $K \subset T$. Therefore for any $i \geq 0$, the point ${X_i}^{-1} j(\mathcal{Q}_\mathcal{K}(X))$ lies in $K \subset T$. Thus we can apply Lemma 2.7 to get $\mathcal{Q}_T(X) = j(\mathcal{Q}_\mathcal{K}(X))$. $\square$

Lemma 5.3. *The admissible lamination of $\mathcal{K}$ is contained in the dual lamination of T:*

$$L_{adm}(\mathcal{K}) \subset L(T)$$

Proof. The admissible lamination $L_{\text{adm}}(\mathcal{K})$ (see §3.5) is defined by all biinfinite words Z in $\mathcal{A}^{\pm}$ such the two half-words Z^+ and Z^- have non-empty domain, and the two domains intersect non-trivially. Thus $\mathcal{Q}_\mathcal{K}(Z^+) = \mathcal{Q}_\mathcal{K}(Z^-)$ is a point in K. Thus by Lemma 5.2 one has $\mathcal{Q}_T(Z^+) = \mathcal{Q}_T(Z^-)$. The latter implies (and is equivalent to) that Z belongs to $L(T)$. $\square$

We sumarize the above discussion in the following commutative diagram:

$$\begin{tikzcd}
& \partial F_N \arrow[dl, "\mathcal{Q}_\mathcal{K}"'] \arrow[dr, two heads, "\mathcal{Q}_T"] & \\
\widehat{T}_\mathcal{K}^{\text{obs}} & & \widehat{T}^{\text{obs}} \\
T_\mathcal{K} \arrow[u, hook] \arrow[rr, "j"] & & \overline{T} \arrow[u, hook]
\end{tikzcd}$$

All the maps in the diagram are equivariant and continuous, where the topology considered on the bottom line is the metric topology.

We can now prove the main result of this paper. Recall from §2.7 that for any basis $\mathcal{A}$ of F_N and T as above the set $\Omega_\mathcal{A} \subset \overline{T}$ denotes the limit set of T with respect to $\mathcal{A}$.

Theorem 5.4. *Let T be an $\mathbb{R}$-tree with very small minimal F_N-action by isometries, and with dense orbits. Let $\mathcal{A}$ be a basis of F_N, and let $K \subset \overline{T}$ be a compact subtree which satisfies $K \cap a_i K \neq \emptyset$ for all $a_i \in \mathcal{A}$. Then the following are equivalent:*

(1) The restriction of the canonical map $j : T_{\mathcal{K}} \to \overline{T}$ to the minimal F_N-invariant subtree $T_{\mathcal{K}}^{min}$ of $T_{\mathcal{K}}$ defines an isometry $j^{min} : T_{\mathcal{K}}^{min} \to T$.

(2) $L(T) \subset L_{adm}(\mathcal{K})$ $\quad$ ($\Longleftrightarrow L(T) = L_{adm}(\mathcal{K})$, by Lemma 5.3)

(3) $\Omega_{\mathcal{A}} \subset K$

Proof. (1) $\Longrightarrow$ (2): By the assumption on j the minimal subtree $T_{\mathcal{K}}^{min} \subset T_{\mathcal{K}}$ is isometric to T. Hence the dual laminations satisfy $L(T) = L(T_{\mathcal{K}}^{min})$, and by Remark 2.5 one has $L(T_{\mathcal{K}}^{min}) = L(T_{\mathcal{K}})$. We now apply Proposition 3.11 to get $L(T_{\mathcal{K}}) \subset L_{\mathrm{adm}}(\mathcal{K})$.

(2) $\Longrightarrow$ (3): By Definition 2.11, a point $Q \in T$ belongs to the limit set $\Omega_{\mathcal{A}}$ if and only if there is a pair of infinite words $(X, Y) \in L^2(T) \subset \partial^2 F(\mathcal{A})$, with initial letters $X_1 \neq Y_1$, which satisfy $Q_T(X) = Q_T(Y) = Q$. By assumption, $L(T)$ is a subset of $L_{\mathrm{adm}}(\mathcal{K})$, so that the reduced words X, Y and $X^{-1} \cdot Y$ are admissible for the system of isometries $\mathcal{K}$. By Definition 4.6, $\{\mathcal{Q}_{\mathcal{K}}(X)\}$ is the domain of X and Y, and thus is contained in K. We deduce from Lemma 5.2 that $j(\mathcal{Q}_{\mathcal{K}}(X)) = \mathcal{Q}_T(X) = Q$, and Q lies in K.

(3) $\Longrightarrow$ (2): Let Z be a biinfinite indexed reduced word in the symbolic lamination $L_{\mathcal{A}}(T)$ defined by the dual lamination $L(T)$ of T (see §2.3). That is to say, $Z = (Z^-)^{-1} \cdot Z^+$, written as a reduced product, and $\mathcal{Q}_T(Z^-) = \mathcal{Q}_T(Z^+)$ is a point $Q \in \Omega_{\mathcal{A}}$. For any $n \in \mathbb{Z}$, we consider the shift $\sigma^n(Z)$ of Z as in Remark 2.4. If u is the prefix of Z^+ of length n (or, if $n < 0$, the prefix of Z^- of length $-n$), then $\sigma^n(Z) = (Z^-)^{-1} u \cdot u^{-1} Z^+$ and $\mathcal{Q}_T(u^{-1}Z^+) = \mathcal{Q}_T(u^{-1}Z^-) = u^{-1}Q$, and this is again a point of $\Omega_{\mathcal{A}}$ and thus contained in K, by hypothesis. Therefore, both Z^+ and Z^- are admissible, and $\mathrm{dom}(Z^+) = \mathrm{dom}(Z^-) = \{Q\}$. Thus Z is an admissible biinfinite word of the system of isometries $\mathcal{K} = (K, \mathcal{A})$, which shows $L(T) \subset L_{\mathrm{adm}}(\mathcal{K})$.

(2) $\Longrightarrow$ (1): Since the dual lamination $L(T)$ is a subset of the admissible lamination $L_{\mathrm{adm}}(\mathcal{K})$, for any pair of distinct infinite words $X, Y \in \partial F(\mathcal{A})$ the equality $\mathcal{Q}_T(X) = \mathcal{Q}_T(Y)$ implies that $X^{-1}Y$ is admissible, and from Definition 4.6 we deduce $\mathcal{Q}_{\mathcal{K}}(X) = \mathcal{Q}_{\mathcal{K}}(Y)$. Thus the map $\mathcal{Q}_{\mathcal{K}} : \partial F_N \to \widehat{T}_{\mathcal{K}}^{\mathrm{obs}}$ factors over the quotient map $\pi : \partial F_N \to \partial F_N / L^2(T)$ (see §2.6) to define an equivariant map $s : \partial F_N / L^2(T) \to \widehat{T}_{\mathcal{K}}^{\mathrm{obs}}$.

As the topology on $\partial F_N / L^2(T)$ is the quotient topology (see §2.6) and as $\mathcal{Q}_{\mathcal{K}}$ is continuous (see Corollary 4.10), the map s is continuous. Since $\varphi : \partial F_N / L^2(T) \to \widehat{T}^{\mathrm{obs}}$ is a homeomorphism (see Theorem 2.10), we deduce that the image of s is an F_N-invariant connected subtree of $\widehat{T}_{\mathcal{K}}^{\mathrm{obs}}$. Therefore the image of s contains the minimal subtree $T_{\mathcal{K}}^{\mathrm{min}}$ of $T_{\mathcal{K}}$.

As a consequence, for any point P in $T_{\mathcal{K}}^{\min}$ there exists an element $X \in \partial F_N$ such that $s(\pi(X)) = \mathcal{Q}_{\mathcal{K}}(X) = P$. From Lemma 5.2 we obtain $j^{\min}(P) = j(\mathcal{Q}_{\mathcal{K}}(X)) = \mathcal{Q}_T(X)$. By definition of the homeomorphism φ, one has $\varphi^{-1}(j^{\min}(P)) = \pi(X)$ and $s(\varphi^{-1}(j^{\min}(P))) = P$. This proves that $j^{\min}$ is injective.

$$\begin{tikzcd}
 & \partial F_N \arrow[d, "\pi"] \arrow[ddl, "\mathcal{Q}_{\mathcal{K}}"'] \arrow[ddr, "\mathcal{Q}_T"] & \\
 & \partial F_N/L^2(T) \arrow[dl, "s"'] \arrow[dr, "\varphi", "\simeq"'] & \\
\widehat{T}_{\mathcal{K}}^{\mathrm{obs}} & & \widehat{T}^{\mathrm{obs}} \\
T_{\mathcal{K}} \arrow[u, hook] \arrow[rr, "j"] & & \overline{T} \arrow[u, hook] \\
T_{\mathcal{K}}^{\min} \arrow[u, hook] \arrow[rr, "j^{\min}"] & & T \arrow[u, hook]
\end{tikzcd}$$

Since j is continuous with respect to the metric topology, since j maps K isometrically, and since $T_{\mathcal{K}} = F_N K$, this implies that $j^{\min}$ is an isometry. $\square$

Recall from §2.7 that the heart $K_{\mathcal{A}} \subset \overline{T}$ denotes the convex hull of the limit set $\Omega_{\mathcal{A}}$ of T with respect to the basis $\mathcal{A}$. We denote by $\mathcal{K}_{\mathcal{A}} = (K_{\mathcal{A}}, \mathcal{A})$ the associated system of partial isometries.

We remark that, in the above theorem, the map $\mathcal{Q}_{\mathcal{K}}$ may fail to be surjective onto $T_{\mathcal{K}}$ if K is too large. And hence, j may fail to be injective even if the limit set $\Omega_{\mathcal{A}}$ is contained in K. This is the reason why we considered the minimal subtree $T_{\mathcal{K}}^{\min}$ of $T_{\mathcal{K}}$. However if K is exactly equal to the heart $K_{\mathcal{A}}$ we get the following corollary.

Corollary 5.5. *Let T be an $\mathbb{R}$-tree with very small minimal F_N-action by isometries, and with dense orbits. Let $\mathcal{A}$ be a basis of F_N, with heart $K_{\mathcal{A}}$. The map $j : T_{\mathcal{K}_{\mathcal{A}}} \to \overline{T}$ is isometric and its image contains T.*

Proof. By definition, for $K = K_{\mathcal{A}}$ the three equivalent conditions of Theorem 5.4 are satisfied.

In the proof of implication (2)⇒(3) of Theorem 5.4, we proved that $\Omega_{\mathcal{A}}$ is in the image of $\mathcal{Q}_{\mathcal{K}}$. In the proof of implication (2)⇒(1), we proved that the image of $\mathcal{Q}_{\mathcal{K}}$ is connected and that j is injective on the image of $\mathcal{Q}_{\mathcal{K}}$.

Therefore $K_{\mathcal{A}}$ is in the image of $\mathcal{Q}_{\mathcal{K}}$, and the map $j : T_{\mathcal{K}_{\mathcal{A}}} \to \overline{T}$ is injective. From the last paragraph of the proof of Theorem 5.4 we deduce that j is isometric. Finally, from the minimality of T we deduce that the image of j contains T. $\square$

6 Applications to geometric trees and limits

In this section we will present some first applications of the main result of this paper, Theorem 5.4, to questions which in part date back to the work of Gaboriau-Levitt [GL95]. It should also be noted that Theorem 5.4 is the basis for the forthcoming papers [Cou08] and [CH08].

Recall that Outer space CV_N is the space of projectivized minimal free simplicial actions of F_N on $\mathbb{R}$-trees. It comes with a natural action by $\mathrm{Out}(F_N)$, and it is in many ways the analogue of Teichmüller space, equipped with its action of the mapping class group. In particular, CV_N has a natural "Thurston boundary" $\partial\mathrm{CV}_N$, which defines a compactification $\overline{\mathrm{CV}}_N = \mathrm{CV}_N \cup \partial\mathrm{CV}_N$ of CV_N. Its preimage $\overline{\mathrm{cv}}_N$, obtained through unprojectivization, consists precisely of all $\mathbb{R}$-trees T with non-trivial minimal very small action of F_N by isometries.

6.1 Geometric trees

There is a special class of group actions on $\mathbb{R}$-trees which play an important role in what is often called the "Rips machine": A minimal $\mathbb{R}$-tree T is called *geometric* if there exists a finite subtree $K \subset T$ and a basis $\mathcal{A}$ of F_N such that the map $j : T_{\mathcal{K}} \to T$ is an isometry. It is proved in [GL95] that in this case for any basis $\mathcal{A}$ one can find such a finite subtree K. For more information about geometric trees regarding the context of this paper see [GL95].

Recall from §2.7 that the heart $K_{\mathcal{A}} \subset T$ denotes the convex hull of the limit set $\Omega_{\mathcal{A}}$ of T with respect to the basis $\mathcal{A}$. We denote by $\mathcal{K}_{\mathcal{A}} = (K_{\mathcal{A}}, \mathcal{A})$ the associated system of partial isometries.

Corollary 6.1. *A very small minimal $\mathbb{R}$-tree T, with isometric F_N-action that has dense orbits, is geometric if and only if, for any basis $\mathcal{A}$ of F_N, the heart $K_{\mathcal{A}}$ is a finite subtree of T.*

Proof. If T is geometric, then by definition there is a finite tree $K \subset T$ such that the map $j : T_K \to T$ is an isometry. Thus condition (1) of Theorem 5.4 is satisfied, and hence condition (3) implies that $K_{\mathcal{A}}$ is a subtree of K, and thus it is finite.

Conversely, if $K_{\mathcal{A}}$ lies in T, the image of the map j defined on $T_{\mathcal{K}_{\mathcal{A}}}$ is contained in $T \subset \overline{T}$, giving a map $j : T_{\mathcal{K}_{\mathcal{A}}} \to T$ which by Corollary 5.5 is isometric. By minimality of T, the map j is onto. □

6.2 Increasing systems of isometries

Let T be a minimal $\mathbb{R}$-tree with a very small action of F_N by isometries, which has dense orbits. As in §5, let $\mathcal{A}$ be a basis of F_N, and for any $n \in \mathbb{N}$ let $K(n)$ be a compact subtree of the metric completion $\overline{T}$ of T, with non-empty intersections $K(n) \cap a_i K(n)$ for all a_i of $\mathcal{A}$. One obtains systems of partial isometries $\mathcal{K}(n) = (K(n), \mathcal{A})$ as in the previous sections.

We will consider sequences $K(n)$ which are increasing, i.e. for all $n \le m$ we assume

$$K(n) \subset K(m) .$$

We can apply Theorem 3.2 to the case $K = K(n)$ and the tree $T_{\mathcal{K}(m)}$, to obtain canonical F_N-equivariant maps

$$j_{m,n} : T_{\mathcal{K}(n)} \to T_{\mathcal{K}(m)}$$

which satisfy $j_{k,m} \circ j_{m,n} = j_{k,n}$, for any natural numbers $n \le m \le k$.

The maps $j_{m,n}$ are length decreasing morphisms, so that the trees $T_{\mathcal{K}(n)}$ converge in the equivariant Gromov-Hausdorff topology (see [Pau88]) to an $\mathbb{R}$-tree T_∞, equipped with an action of F_N by isometries. Alternatively, one can pass to the direct limit space defined by the system of maps $j_{m,n}$, which inherits from the $T_{\mathcal{K}(n)}$ a canonical pseudo-metric as well as an action of F_N by (pseudo-)isometries. One then defines T_∞ as the canonically associated metric quotient space. Both, arc-connectedness and 0-hyperbolicity carry over in those transitions, so that T_∞ is indeed an $\mathbb{R}$-tree with isometric F_N-action.

The minimal F_N-invariant subtrees $T^{\min}_{\mathcal{K}(n)} \subset T_{\mathcal{K}(n)}$ and $T^{\min}_\infty \subset T_\infty$ define points in the closure $\overline{\mathrm{cv}}_N$ of unprojectivized Outer space cv_N (compare [CHL-II] and the references given there). The sequence of trees $T^{\min}_{\mathcal{K}(n)}$ converges in $\overline{\mathrm{cv}}_N$ to the tree $T^{\min}_\infty$.

The maps $j_{m,n}$ also converge to F_N-equivariant maps $j_{\infty,n} : T_{\mathcal{K}(n)} \to T_\infty$ that satisfy $j_{\infty,m} \circ j_{m,n} = j_{\infty,n}$.

We consider the increasing union of the $K(n)$, and we define $K(\infty)$ to be its closure in $\overline{T}$,

$$K(\infty) = \overline{\bigcup_{n\in\mathbb{N}} K(n)} ,$$

provided with the induced system $\mathcal{K}(\infty) = (K(\infty), \mathcal{A})$ of partial isometries. We always assume that $K(\infty)$ is compact.

Using that $K(n) \subset K(\infty)$, we can apply again Theorem 3.2 to get F_N-equivariant, length decreasing morphisms:

$$j_{0,n} : T_{\mathcal{K}(n)} \to T_{\mathcal{K}(\infty)}$$

These maps converge to an F_N-equivariant length decreasing map $j_{0,\infty} : T_\infty \to T_{\mathcal{K}(\infty)}$. This map continuously extends to the metric completions $\overline{\jmath}_{0,\infty} : \overline{T}_\infty \to \overline{T}_{\mathcal{K}(\infty)}$.

For each $n \in \mathbb{N}$, the map $j_{0,n}$ restricts to an isometry on $K(n)$, and thus $j_{0,\infty}$ restricts to an isometry on the union of the $K(n)$, which extends to an isometry from $K(\infty) \subset \overline{T}_\infty$ onto its image in $T_{\mathcal{K}(\infty)}$. Applying Theorem 3.2 again, to the inverse of this isometry, we get an F_N-equivariant length decreasing morphism $j_{\infty,0} : T_{\mathcal{K}(\infty)} \to \overline{T}_\infty$.

By construction, the restrictions of the maps $j_{0,\infty}$ and $j_{\infty,0}$ to each of the $K(n)$ are isometries which are inverses of one another. Thus the map $\overline{\jmath}_{0,\infty} \circ j_{\infty,0}$

is length decreasing and restricts to the identity on $\bigcup_{n\in\mathbb{N}} K(n)$ and thus on $K(\infty)$. Using Theorem 3.2, we see that it is an isometry on all of $T_{\mathcal{K}(\infty)}$. This shows

$$\lim_{n\to+\infty} T_{\mathcal{K}(n)} = T_\infty \subset T_{\mathcal{K}(\infty)}$$

and thus

$$\lim_{n\to+\infty} T^{\min}_{\mathcal{K}(n)} = T^{\min}_{\mathcal{K}(\infty)} .$$

As a direct consequence of Theorem 5.4 one now derives:

Corollary 6.2. *Let T be a minimal $\mathbb{R}$-tree with a very small action of F_N by isometries, which has dense orbits. Let $\mathcal{A}$ be a basis of F_N. For any $n \in \mathbb{N}$, let $K(n)$ be a compact subtree of $\overline{T}$ with non-empty intersections $K(n) \cap a_i K(n)$, for all a_i of $\mathcal{A}$. Let $\mathcal{K}(n) = (K(n), \mathcal{A})$ be the induced systems of isometries. Let $K(\infty)$ be the closure of the increasing union of the $K(n)$, and assume that $K(\infty)$ is compact.*

Then the minimal trees $T^{min}_{\mathcal{K}(n)}$ converge in $\overline{cv}_N$ to T if and only if $K(\infty)$ contains $\Omega_{\mathcal{A}}$. □

An application of this corollary is the following sharpening of a classical result of Gaboriau-Levitt [GL95], who showed that every $T \in \overline{\mathrm{cv}}_N$ can be approximated by a sequence of geometric $T_{\mathcal{K}(n)}$, i.e. each $K(n)$ is a finite subtree of T.

Corollary 6.3. *For every very small minimal $\mathbb{R}$-tree T, with isometric F_N-action that has dense orbits, there exists a sequence of finite subtrees $K(n)$ of uniformely bounded diameter, such that:*

$$T = \lim_{n\to\infty} T_{\mathcal{K}(n)}$$

Proof. It is well known [GL95] that the number of branch points in T is a countable set $\mathcal{P}$ that is dense in every segment of T. It suffices to consider the countable subfamily $(P_n)_{n\in\mathbb{N}} = \mathcal{P} \cap K_{\mathcal{A}}$ and to define $K(n)$ as convex hull of the set $\{P_1, \ldots, P_n\}$. Since the heart $K_{\mathcal{A}}$ is the convex hull of the limit set $\Omega_{\mathcal{A}}$, the claim is a direct consequence of Corollary 6.2. □

6.3 Approximations by simplicial trees

An algebraic lamination L is said to be *closed by diagonal leaves*, if for any leaves (X, X') and (X', X'') in L one either has $X = X''$, or (X, X'') is again a leaf in L. We remark that, if T is an $\mathbb{R}$-tree with a minimal action of F_N by isometries that has dense orbits, it follows from §2.6 that the dual lamination $L(T)$ of T is closed by diagonal leaves. Also, for a system of isometries $\mathcal{K}$ with independent generators, we deduce from §3.5 and §4.2 that the admissible lamination $L_{\mathrm{adm}}(\mathcal{K})$ is closed by diagonal leaves.

An algebraic lamination L is said to be *minimal up to diagonal leaves* if it does not contain a proper non-trivial sublamination that is closed by diagonal leaves.

Corollary 6.4. *Let T be an $\mathbb{R}$-tree with a minimal, very small action of F_N that has dense orbits. Let $\mathcal{A}$ be a basis of F_N. If a compact subtree $K \subset \overline{T}$ does not contain $\Omega_{\mathcal{A}}$, and if the dual lamination $L(T)$ is minimal up to diagonal leaves, then the approximation tree $T_{\mathcal{K}}^{min}$ is free simplicial (i.e. it belongs to the unprojectivized Outer space cv_N rather than to its boundary ∂cv_N).*

Proof. From Proposition 3.11 we know that the dual lamination $L(T_{\mathcal{K}})$ of $T_{\mathcal{K}}$ is a sublamination of the admissible lamination $L_{\mathrm{adm}}(\mathcal{K})$. The lamination $L_{\mathrm{adm}}(\mathcal{K})$ is closed by diagonal leaves and is a sublamination of $L(T)$, by Lemma 5.3. Since $\Omega_{\mathcal{A}}$ is not a subset of K, Theorem 5.4 implies that the admissible lamination $L_{\mathrm{adm}}(\mathcal{K})$ is a strict sublamination of $L(T)$.

From the minimality of $L(T)$ up to diagonal leaves we deduce that $L_{\mathrm{adm}}(\mathcal{K})$ and $L(T_{\mathcal{K}})$ are empty, so that (compare §2.4) the action of F_N on $T_{\mathcal{K}}^{\min}$ is free and discrete. $\square$

This corollary indicates that the resolution of an arbitrary $\mathbb{R}$-tree with isometric G-action, for more general groups G, via systems of partial isometries on a finite tree, as promoted by the Rips machine, may yield directly a simplicial tree, i.e. without having to go through further iterations in Rips' procedure.

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